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SPECIAL ISSUE ARTICLE

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Information, trade and incomplete markets

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Abstract The no-trade result of Milgrom and Stokey, J Econ Theory 26:17–27 (1982), states that if rational traders begin with an ex-ante Pareto optimal allocation then the arrival of information cannot generate trade. This paper allows traders to trade before and after the arrival of information. If there are enough securities to hedge against all payoff relevant risk, then the preinformation-arrival allocation is Pareto optimal and information arrival has no effect. This no-retrade result is the competitive analog of the no-trade result of Milgrom and Stokey (1982). However, information generically generates trade when markets are state-contingent incomplete.

Keywords Trade · Incomplete markets · Risk sharing

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1 Introduction

A standard interpretation of the no-trade theorem in the fundamental paper by Milgrom and Stokey (1982) is that the arrival of new private information cannot generate trade between rational traders in an ongoing security market. The usual

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intuition offered for this theorem is as follows. Suppose that traders' initial security holdings are Pareto optimal and that some traders receive new information. If this new information generates trade then the trade must be for speculative purposes. But any trader who has not received the information would not want to trade as each such trader would know that he is being taken advantage of by the informed traders. [1]

This reasoning is game theoretic; it is not based on the decision making that is usually employed in competitive markets. Under some circumstances the no-trade result applies to rational expectations equilibrium, but the logic above does not apply. There are two reasons for this. First, to make the argument above carefully requires setting up a game and employing a game theoretic equilibrium concept. This requires knowledge assumptions that are irrelevant in competitive equilibrium analysis. In a competitive market no trader reasons about, or needs to reason about, the knowledge, payoffs or rationality of any other trader. In a rational expectations equilibrium every trader makes rational inferences about relevant information from market statistics. But this requires only rationality and knowledge of the equilibrium relationship; it does not require higher order knowledge assumptions. [2] Second, if one attempts to employ the reasoning above one typically does not get a rational expectations equilibrium even in settings in which traders might trade for risk sharing purposes as well as for speculation. Blume and Easley (1990) show that for a broad class of classical economies, there is no game whose Bayes Nash equilibrium is a rational expectations equilibrium unless stringent restrictions are placed on the distribution of information. These restrictions exclude economies in which informed traders can take advantage of uninformed traders.

Milgrom and Stokey (1982) also provide a non-game theoretic analysis in which they show that if beliefs satisfy a restriction, and if traders begin with a Pareto optimal allocation of state contingent consumptions, then the arrival of public information does not generate rational expectations equilibrium trade. Kreps (1977) provides a similar rational expectations equilibrium result as a No-Speculation theorem. Judd, Kubler, and Schmedders (2003) show in a Lucas asset pricing model with infinitely lived assets forming dynamically complete markets that the volume of trade is zero beyond the first period.

The reason that no trade occurs is that, under the belief restriction, the arrival of information preserves equality of marginal rates of substitution across traders. The allocations are assumed to be ex ante Pareto optimal, so consumers begin with a common marginal rate of substitution. Information arrival changes marginal rates of substitution, but since traders have a common interpretation of the signal, equality of marginal rates of substitution across traders is preserved and the original allocation remains an equilibrium allocation. Thus there is no trade even though beliefs have changed.

This result has two key assumptions: the Pareto optimality of the original allocation and the belief restriction. The Pareto optimality of the initial allocation is usually interpreted as arising from a previous round of trade on competitive markets.

[1] For discussions of No-Trade for speculative purposes results focusing on their game theoretic origin see Fudenberg and Tirole (1991) and Rubinstein and Wolinsky (1990).

[2] Dutta and Morris (1997) provide a common knowledge foundation for rational expectations equilibrium. As with similar foundations for Nash equilibrium this reasoning provides sufficient, but not necessary conditions.

The first welfare theorem implies that if the markets are complete then any competitive equilibrium allocation will be Pareto optimal. So in an ongoing complete market, in which there are no new liquidity reasons for trade (no exogenous changes in endowments), the initial allocation at each trading date will be Pareto optimal. Numerous authors, including Milgrom and Stokey (1982), note that completeness of markets matters for this claim. We show that if markets are state-contingent incomplete, then for the generic economy the arrival of new information does generate trade. In this theorem we do not impose any restrictions on beliefs. Even if agents have common priors, the arrival of new information generically generates trade in state-contingent incomplete markets economies. This occurs because with incomplete markets the arrival of new information creates new insurance opportunities.

The belief restriction is that consumers share an understanding of the information generating process. In Bayesian terms a sufficient restriction is that they have a common likelihood function. In a standard statistical problem assuming a correct, and therefore common, likelihood function is natural. It is less natural in a consumer decision problem. Suppose one begins with consumers who have preferences over random consumptions defined on a state space consisting of the payoff relevant states and the signals. Then we typically derive a consumer's beliefs over the space of signals and states as well as his utility function from the expected utility theorem. This approach places no restrictions on beliefs. We show that without the belief restriction the arrival of new information generically generates trade even when consumers begin with a Pareto optimal allocation. [3]

The no-trade result is often cited as a reason why noise traders are needed to generate trade or why behavioral, not fully rational, traders must be present in markets which experience significant volume of trade. There are good reasons to consider noise or behavioral traders, but our analysis shows that the argument that they must be there because we see more trade than can be accounted for by liquidity reasons is not as simple as it usually made out to be. It requires a restriction on how traders interpret information that goes beyond what is usually meant by an assumption of rationality. It also requires complete markets. We do not believe that either of these assumptions are descriptive.

The paper is organized as follows. The second section presents the economic model used throughout the paper. The third section provides a simple example to illustrate the main ideas. The fourth and fifth sections analyze the role of public and asymmetric information, respectively, in generating trade in securities markets relative to the market structure. The final section concludes.

2 The model

We analyze a three period, one good economy represented in a tree structure. Node ξ_0 represents period $t = 0$, nodes $Z = \{\xi_1, \dots, \xi_L\}$ represent possible signals in period $t = 1$. States $\Theta = \{1, \dots, S\}$ represent possible payoff-relevant states in period $t = 2$.

[3] For an extensive analysis of how heterogeneity in prior beliefs affects no-trade type results in a variety of settings see Morris (1994).

At time $t = 0$ individuals trade J state-contingent securities to hedge against state-contingent endowment risk. At time $t = 1$, after signals arrive, individuals can retrade the securities. At time $t = 2$ the securities pay off in the single physical good available in the economy. Payoffs to securities depend only on the $t = 2$ state. They are described by a $S \times J$ matrix of security payoffs V whose (s, j)th element V_s^j is the payoff of security j in state s. After the securities pay off, individuals consume the sum of their endowment in that state and the payoffs that result from their portfolio choice. Consumption occurs only at $t = 2$.

The security price vector at time $t = 0$ is denoted $q(\xi_0)$. The security price vector at time $t = 1$ if signal ξ_l occurs is denoted $q(\xi_l)$. We assume that there are no redundant securities.

Assumption 1 *The matrix of security payoffs V has rank J.*

There are I individuals indexed by h. Individuals are described by utility functions, beliefs and endowments. Agent h's consumption in state s when signal ξ occurs is denoted by $x_s^h(\xi)$ and h's (Bernoulli) utility of this consumption is denoted by $u^h(x_s^h(\xi))$. We assume that utility functions are infinitely differentiable, strictly increasing, exhibit strict risk aversion, and satisfy an Inada condition.

Assumption 2 *For all h, $u^h : \mathbb{R}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}_+$, infinitely differentiable on $\mathbb{R}_{++}$, strictly increasing, strictly concave and satisfies the Inada condition: $\lim_{x \to 0} u^{h'}(x) = \infty$.*

Agent h's beliefs are described by a strictly positive joint probability distribution over signals and states which is denoted by $\pi^h : Z \times \Theta \to (0, 1)$. Her unconditional expected utility of consumption is therefore:

$$U^h(x^h) = \mathbb{E}^h(u^h(x^h)) = \sum_{s,l} \pi^h(\xi_l, s) u^h(x_s^h(\xi_l))$$

And her expected utility of consumption, conditional on signal ξ is denoted by:

$$U_\xi^h(x^h) = \mathbb{E}^h(u(x^h)|\xi) = \sum_s \pi^h(s|\xi) u^h(x_s^h(\xi))$$

where $\pi^h(s|\xi)$ is the conditional probability of s given ξ.

The following restriction on agents' beliefs will be needed in the statement of the main theorem. Let $\pi^h(\xi|s)$ be the conditional probability of ξ given s.

Assumption 3 *For all ξ, h, h', s, s',*

$$\frac{\pi^h(\xi|s)}{\pi^h(\xi|s')} = \frac{\pi^{h'}(\xi|s)}{\pi^{h'}(\xi|s')}$$

This assumption states that likelihood ratios are equated across agents. If, as in Milgrom and Stokey (1982), agents' beliefs are concordant (namely, $\pi^h(\xi|s) = \pi^{h'}(\xi|s)$), then this assumption is satisfied. The assumption of belief concordance implies that agents interpret the meaning of signals in similar ways.

While agents' endowments are stochastic, we assume that they are strictly positive and that they are not affected by signals. Formally,

Assumption 4 *$\omega^h \in \mathbb{R}^{SL}_{++}$ and for all ξ, ξ', h, s: $\omega^h_s(\xi) = \omega^h_s(\xi')$.*

The space Ω of endowments with this property is a SI-dimensional subspace of $\mathbb{R}^{SLI}_{++}$. For simplicity, we write $\Omega = \mathbb{R}^{SI}_{++}$.

Definition 1 *An economy $\mathcal{E}$ is a collection of endowments, utilities, beliefs and a matrix of security payoffs $\{\omega, u, \pi, V\}$.*

Note that in our economies signals play only an informational role. Utility functions and endowments are not affected by signals. So Pareto optimal allocations are not signal dependent. Alternatively, if utilities or endowments are signal dependent then Pareto optimal allocations may be signal dependent also.

3 An example

In this section we construct a simple example to provide some intuition for our results. In this example there are two signals and three states: $Z = \{\xi_1, \xi_2\}$ and $\Theta = \{1, 2, 3\}$.

Suppose that prior to the arrival of signals, state contingent consumption allocations are Pareto optimal. This could occur, for example, if individuals consumption plans were the result of trade at time 0 of state contingent complete securities. Evaluated at Pareto optimal plans, individuals' marginal rates of substitution between consumption in various states are equal. So for any two states s and t, and any two individuals h and j, we have

$$\frac{\pi^h(s)u^{h'}(x^h_s)}{\pi^h(t)u^{h'}(x^h_t)} = \frac{\pi^j(s)u^{j'}(x^j_s)}{\pi^j(t)u^{j'}(x^j_t)}$$

where $\pi^h(s) = \sum_l \pi^h(s, \xi_l)$ is h's prior probability of state s.

Suppose that at time $t = 1$ all traders observe signal ξ_l. Traders observe it either because it is public information or because it is fully revealed in a rational expectations equilibrium. In either case, the relevant probabilities for individual's decision problems are now conditional probabilities on states given the signal. By Bayes rule these are

$$\pi^h(s|\xi_l) = L^h(\xi_l, s)\pi^h(s)$$

where $L^h(\xi_l, s) = \pi^h(\xi_l|s)/\pi^h(\xi_l)$ is the likelihood according to individual h of signal ξ_l given state s.

Suppose that after the information arrival individuals trade a complete set of Arrow securities. That is, the security payoff matrix, V, is the 3×3 identity matrix. Let y^h_s be the resulting equilibrium consumption plans. These consumptions are characterized by feasibility and by equality of updated marginal rates of substitution.

$$\frac{\pi^h(s)L^h(\xi_l, s)u^{h'}(y^h_s)}{\pi^h(t)L^h(\xi_l, t)u^{h'}(y^h_t)} = \frac{\pi^j(s)L^j(\xi_l, s)u^{j'}(y^j_s)}{\pi^j(t)L^j(\xi_l, t)u^{j'}(y^j_t)}$$

By Assumption 3, individuals have common likelihood ratios. Thus the original consumption plans x_s^h remain solutions. These plans were feasible so they are equilibrium plans. Since equilibrium consumption plans do not change, the arrival of the signal need not generate security trade.

The conclusion that the arrival of information does not generate trade is independent of the set of assets traded at time $t = 1$. To see this suppose that there are only two assets. Asset one pays off one unit of the good in states 1 and 2 and 0 in state 3. Asset two pays off one unit of the good in state 3 and 0 in states 1 and 2. Now consumptions plans y_s^h are equilibrium plans if they are feasible and if for each pair of traders i and j

$$\frac{\pi^h(1)L^h(\xi_l, 1)u^{h'}(y_1^h) + \pi^h(2)L^h(\xi_l, 2)u^{h'}(y_2^h)}{\pi^h(3)L^h(\xi_l, 3)u^{h'}(y_3^h)} \tag{1}$$

$$= \frac{\pi^j(1)L^j(\xi_l, 1)u^{j'}(y_1^j) + \pi^h(2)L^j(\xi_l, 2)u^{j'}(y_2^j)}{\pi^j(3)L^j(\xi_l, 3)u^{j'}(y_3^j)} \tag{2}$$

Writing each trader's marginal rate of substitution as the sum of two fractions, using the equality of marginal rates of substitution at x_s^h and the common likelihood ratios it is easy to see that the original consumption plans remain an equilibrium. So even if markets are incomplete, the arrival of information need not generate trade from a Pareto optimal allocation.

However, if the original allocation was obtained via trade of an incomplete set of securities it need not be Pareto optimal. If at time $t = 0$, before signals arrive, individuals trade the securities above then no-retrade equilibrium consumption plans, x_s^h, solve

$$\frac{\pi^h(1)u^{h'}(x_1^h) + \pi^h(2)u^{h'}(x_2^h)}{\pi^h(3)u^{h'}(x_3^h)} = \frac{\pi^j(1)u^{j'}(x_1^j) + \pi^j(2)u^{j'}(x_2^j)}{\pi^j(3)u^{j'}(x_3^j)}$$

These plans remain part of an equilibrium after signal ξ_l arrives only if they solve equation 2. This can occur only if the incomplete markets equilibrium consumption plans, x_s^h, are Pareto optimal or if the signal does not change any trader's beliefs. In the subsequent sections we assume that for at least one trader signals are meaningful. For the generic economy (in the space of endowments), incomplete markets are not rich enough to permit all of the risk sharing that individuals desire and thus equilibrium consumption plans are not Pareto optimal. So if signals are meaningful, and markets are incomplete, then the arrival of information generates trade in the generic economy.

4 Public information

In this section, we investigate economies in which information is revealed publicly at the intermediate stage. There are two reasons to consider this case before considering private information. First, it allows us to clearly show the conditions under which the arrival of information does or does not generate trade when issues of being taken advantage of by better informed traders are not relevant. Second, if information is initially private, and one employs a rational expectations equilibrium concept, then for many economies the resulting equilibrium will be a public information equilibrium.

We assume the following timing of trading opportunities and information revelation. At time 0, before any information is revealed, economic agents trade the real securities described by V. At time 1 a signal is revealed publicly to all agents who use it to update their beliefs about the likelihood of time 2 states. Markets reopen, and agents retrade the securities. At time 2, a state is realized, securities pay off and consumption occurs. This sequence of trades and the resulting equilibrium allocation of goods, securities and equilibrium prices is captured by the usual Arrow–Debreu equilibrium concept. Here, (x, z, q) denotes the vector $(x^1, .., x^I, z^1, .., z^I, q_1, .., q_J) \in \mathbf{R}^{ISL+I(1+L)J+(1+L)J}$.

Definition 2 *Given an economy $\mathcal{E}$, an equilibrium is a collection (x, z, q) such that*

(a) $(x^h, z^h) \in \arg\max U^h(x^h)$ *subject to*

$$\begin{aligned} q(\xi_0)z^h(\xi_0) &= 0 \\ q(\xi)z^h(\xi) &= q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z \\ x^h(\xi) - \omega(\xi) &= Vz^h(\xi) \text{ for all } \xi \in Z \end{aligned}$$

(b) $\sum_{h=1}^{I} z^h(\xi_0) = 0, \sum_{h=1}^{I} z^h(\xi) = 0$ *for all* $\xi \in Z$

The first two lines in the budget constraint reflect the assumption that times 0 and 1 there are no endowments and no consumption; only securities are traded. The third line requires agents to consume the net dividend on their security holdings plus their endowment in period 2. The second condition requires securities market clearing at each date.

Whether markets are complete or incomplete is critical for the existence of re-trade in this equilibrium. The relevant notion of market completeness in our framework is state-completeness.[4]

Definition 3 *Markets are state-contingent complete, or Θ - complete, if there are S non-redundant securities.*

State-contingent complete markets may or may not be complete in the usual sense. For our economy state-completeness of markets implies completeness in the usual sense if for example $J \geq L$ and if in equilibrium the following matrix has rank L:

$$\begin{bmatrix} q_1(\xi_1) & \dots & q_J(\xi_1) \\ \dots & \dots & \dots \\ q_1(\xi_L) & \dots & q_J(\xi_L) \end{bmatrix}$$

[4] If utilities or endowments are signal dependent then state-contingent completeness alone would not be sufficient to imply no-retrade.

An equilibrium involves no retrade if:

$$z^h(\xi_0) = z^h(\xi) \text{ for all } \xi, h$$

Our first result is that in economies with state-contingent complete markets there is no retrade if and only if agents agree about how to interpret signals, i.e., their beliefs satisfy Assumption 3.

This result recasts the Milgrom and Stokey (1982) no-trade result in a market setting. Milgrom and Stokey's argument relies on an assumption that agents begin with an ex-ante Pareto optimal allocation and trade on Θ - complete markets once information arrives. In our economy, agents acquire securities on Θ - complete markets before information arrives. Θ - completeness does not guarantee market completeness and therefore the resulting consumption plans after the first round of trade need not be ex- ante Pareto optimal. Our analysis shows that the consumption plans resulting from trade of Θ - complete securities cannot be improved upon once information arrives, so no retrade occurs, if and only if beliefs satisfy Assumption 3.

Theorem 4 *Suppose Assumptions 1, 2 and 4 are satisfied. Consider an economy $\mathcal{E}$ with Θ - complete markets. Then Assumption 3 is satisfied if and only if there exists a no retrade equilibrium for $\mathcal{E}$.*

All proofs are given in the Appendix.

We now investigate agents' incentive to retrade when markets are incomplete. When the number of signals or states is too few, it is easy to construct examples of economies in which no retrade occurs, regardless of assumptions about market completeness or belief concordance. So, in the statement and proof of the next theorem, we impose the following assumption to ensure that the market and information structure is sufficiently rich.

Assumption 5 (a) $S \geq 3, L \geq 2, I \geq 2$
(b) *For all states $s \in \Theta$, there exists a security k such that $V_s^k > 0$*
(c) *There exists at least one agent h such that the matrix $\left\{\pi^h(\xi|.)\right\}_{\xi \in Z}$ is of full rank*
(d) *There exists a security that pays off in one state only.*

Assumption 5(b) implies that agents can transfer income to every state. We ignore states under autarchy since they add nothing to the analysis. Assumption 5(c) means that at least one agent believes she can impute distinct meanings to distinct signals. Assumption 5(d) is a technical condition used to simplify the proof of the following theorem.

Theorem 5 *Suppose Assumptions 1 through 5 are satisfied. Consider an economy $\mathcal{E}$ with Θ - incomplete markets such that $L \geq S - 1$. There exists a set $\Omega^* \subset \Omega$ whose complement has measure zero such that if $\omega \in \Omega^*$ any equilibrium must involve retrade.*

This theorem remains valid if we assume more generally an incomplete markets economy with multiple consumption goods in each state that are traded on spot markets. Indeed, the key feature in the proof is that when markets are incomplete, agents' marginal rates of substitution differ when their endowments lie in the set will null complement Ω^*. This property remains valid when there are many goods.

5 Asymmetric information

We now suppose that signals are not public. The timing of events is identical to that described in the previous section except that now agents' information structures may differ. Associate with each agent an information partition Z^h of the signal space. Let Y be the meet of these partitions. We next define the relevant equilibrium concept for our economy with asymmetric information.

Definition 6 *Given an economy $\mathcal{E}$, an equilibrium with asymmetric information is a collection (x, z, q) such that*

(a) $(x^h, z^h) \in \arg\max U^h$ *subject to*

$$\begin{aligned} q(\xi_0)z^h(\xi_0) &= 0 \\ q(\xi)z^h(\xi) &= q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z \\ z^h(\xi) &= z^h(\xi') \text{ for all } \xi, \xi' \in \sigma \in Z^h \\ x^h(\xi) - \omega^h(\xi) &= Vz^h(\xi) \text{ for all } \xi \in Z \end{aligned}$$

(b) $\sum_{h=1}^{I} z^h(\xi_0) = 0,\ \sum_{h=1}^{I} z^h(\xi) = 0,\ \sum_{h=1}^{I} x^h(\xi) - \omega^h(\xi) = 0$ *for all* $\xi \in Z$.

Agents' decisions are required to be measurable with respect to their information partitions. These partitions can be interpreted as reflecting individuals private information in an economy in which individuals do not make inferences from market statistics. Alternatively, if a rational expectations equilibrium exists, they can be interpreted as rational expectations equilibrium partitions. In any case, in an equilibrium individuals partition information in some way and we only require measurability of actions with respect to whatever they know. We believe that this is a natural condition, and it is satisfied in a revealing rational expectations equilibrium. So one application of our theorems is to Radner (1979) economies in which the equilibrium is fully revealing.

There does not exist a direct analog to Theorem 4 in state-complete asymmetric information economies. However, one can easily prove the following

Theorem 7 *Suppose Assumptions 1,2 and 4 are satisfied. Consider an economy $\mathcal{E}$ where markets are Θ - complete. If Assumption 3 is satisfied, there exists a no retrade equilibrium with asymmetric information for any information structure.*

Note that a converse to this theorem does not exist in general. If, for example, all agents have the coarsest possible information partition, then it is clear that a no retrade equilibrium with asymmetric information does not imply any restriction on beliefs.

In asymmetric information economies where markets are Θ - incomplete, we require an analog to Assumption 5. Formally,

Assumption 6 (a) $S \geq 3, L \geq 2, I \geq 2$

(b) *For all states $s \in \Theta$, there exists a security k so that $V_s^k > 0$*

(c) *There exists at least one agent h so that the matrix $\left\{\pi^h(\gamma|.)\right\}_{\gamma \in Y}$ is of full rank, where*

$$\pi^h(\gamma|s) = \sum_{\xi \in \gamma} \pi^h(\xi|s)$$

(d) *There exists a security that pays off in one state only.*

We also reformulate Assumption 3 for asymmetric information economies.

Assumption 7 *For all $\gamma \in Y, h, h', s, s'$,*

$$\frac{\pi^h(\gamma|s)}{\pi^h(\gamma|s')} = \frac{\pi^{h'}(\gamma|s)}{\pi^{h'}(\gamma|s')}$$

A natural restriction on beliefs that implies Assumption 7 is the belief concordance assumption, namely that $\pi^h(\xi|s) = \pi^{h'}(\xi|s)$.

Theorem 8 *Suppose Assumptions 1, 2, 4, 6 and 7 are satisfied. Consider an economy $\mathcal{E}$ where markets are Θ - incomplete and where $\#Y \geq S - 1$. There exists a set $\Omega^* \subset \Omega$ whose complement has measure zero such that if $\omega \in \Omega^*$ then any equilibrium with asymmetric information must involve retrade.*

The revelation of information through information partitions can be given various interpretations. The partitions can be simply those induced directly by observation of private signals. Alternatively, they can be rational expectations equilibrium partitions which incorporate both private information and information revealed through prices.

6 Concluding comments

The common wisdom associated with observing trade in a securities market after a news event is that some traders are trading for liquidity reasons, or that information is ambiguous and traders disagree about it's meaning, or that noise traders are present. This intuition has it's theoretical origin in the celebrated paper of Milgrom and Stokey (1982) but it ignores a fundamental role of financial markets: namely, to allow risk sharing among risk averse traders. This paper gives traders non-trivial motives for ongoing risk sharing by assuming that markets are incomplete. If traders endowment levels are arbitrary and they are allowed to trade before and after the arrival of information, then the result of the no-trade theorem can be reformulated as a no retrade theorem. If traders agree about the meaning of information, and markets are state-contingent complete, then the original portfolio of securities that agents traded to before the arrival of information is still optimal after the arrival of information. However, if markets are state-contingent incomplete, then the arrival of information will generate trade, even if information is public and all traders agree on it's meaning. This is because the arrival of information gives traders new risk-sharing opportunities. Thus observing frequent trade or an excessively high trade volume need not imply that investors interpret information differently or that some investors are irrational.

7 Appendix

7.1 Proof of Theorem 4

For the proof of Theorem 4, we will need the following two lemmas.

Definition 9 *Given an economy $\mathcal{E}$, an sequential equilibrium is collection* $(x(\xi_0), x, z, q)$ *where* $x = (x(\xi))_{\xi \in Z} \in \mathbb{R}^{ISL}$ *such that for all* $h \in \{1, .., I\}$:

(a) $(x^h(\xi_0), z^h(\xi_0)) \in \arg\max \sum_{s \in \Theta} \pi^h(s) u^h(x_s^h)$ *subject to*

$$q(\xi_0) z^h(\xi_0) = 0$$
$$x^h(\xi_0) - \omega^h(\xi_0) = V z^h(\xi_0)$$

(b) *For all* $\xi \in Z$, $((x^h(\xi), z^h(\xi)) \in \arg\max \sum_{s \in \Theta} \pi^h(s|\xi) u^h(x_s^h)$ *subject to*

$$q(\xi) z^h(\xi) = q(\xi) z^h(\xi_0)$$
$$x^h(\xi) - \omega^h(\xi) = V z^h(\xi)$$

(c) *Markets clear:* $\sum_{h=1}^{I} z^h(\xi) = 0$, $\sum_{h=1}^{I} x^h(\xi) - \omega^h(\xi) = 0$ *for all* $\xi \in Z \cup \{\xi_0\}$.

The proofs of the following lemmas are omitted.

Lemma 10 *Suppose that* $(x(\xi_0), x, z, q)$ *is a no retrade sequential equilibrium. Then,* (x, z, q) *is a no retrade equilibrium.*

Lemma 11 *Suppose that* (x, z, q) *is a no retrade equilibrium. Then, there exist* $(x(\xi_0))$ *so that* $(x(\xi_0), x, z, q)$ *is a no retrade sequential equilibrium.*

Proof of Theorem 4 (Sufficiency) Suppose that $(x^h(\xi_0), z^h(\xi_0))_{h \in I}$ solves part (a) of a sequential equilibrium [whose solution always exists. See for example Theorem 10.5 in Magill and Quinzii (1998)]. Then, for all h, there exists a vector of multipliers $\lambda^h(\xi_0) \in \mathbb{R}_{++}$ so that $\sum_{s \in \Theta} \pi^h(s) u^{h\prime}(x_s^h(\xi_0)) V_s^k = \lambda^h(\xi_0) q_k(\xi_0)$ for all h, k.

Since markets are Θ - complete there exists a unique $\pi \in \mathbb{R}_{++}^S$ so that $q(\xi_0) = \pi V$. The above FOCs imply that:

$$\sum_{s \in \Theta} \left[-\pi_s \lambda^h(\xi_0) + \pi^h(s) u^{h'}(x_s^h) \right] V_s^k = 0 \text{ for all } h, k \tag{3}$$

Set $\delta_s^h = -\pi_s \lambda^h(\xi_0) + \pi^h(s) u^{h'}(x_s^h)$ for all $s \in \Theta$ and all h. So equation 3 can be rewritten as $\delta V = 0$. Since V is full rank S, this implies that the one and only solution is $\delta = 0$, which in turn implies that:

$$\pi^h(s) = \frac{\pi_s \lambda^h(\xi_0)}{u^{h'}(x_s^h)} \text{ for all } h, s \tag{4}$$

After information is revealed, agents solve the part (b) of the sequential equilibrium problem. We now construct an equilibrium where the solution to this problem is such that $z^h(\xi) = z^h(\xi_0)$ for all $h, \xi \in Z$ and $x^h(\xi) = x^h$ for all $h, \xi \in Z$. Assumption 3 implies the existence of a function g such that, for all s: $g(h, \xi) = \frac{\pi^1(\xi|s)}{\pi^h(\xi|s)}$ for all $\xi \in Z$. By Bayes Rule: $\pi^h(\xi|s) = \pi^h(s|\xi) \frac{\pi^h(\xi)}{\pi^h(s)}$ for all $\xi \in Z$. So that

$\pi^1(s|\xi)\frac{\pi^1(\xi)}{\pi^1(s)} = g\,(h,\xi)\,\pi^h(s|\xi)\frac{\pi^h(\xi)}{\pi^h(s)}$ for all $\xi \in Z$. Replacing π_s^h from equation 4 in this expression implies:

$$\pi^1(s|\xi)u^{1'}(x_s^1)\left(\frac{\pi^1(\xi)}{\lambda^1(\xi_0)}\right) = \pi^h(s|\xi)u^{h'}(x_s^h)\left(\frac{g\,(h,\xi)\,\pi^h(\xi)}{\lambda^h(\xi_0)}\right) \text{ for all } \xi \in Z$$

Set $\lambda^1(\xi) = \left(\frac{\pi^1(\xi)}{\lambda^1(\xi_0)}\right)^{-1}$ for all $\xi \in Z$ and set $\lambda^h(\xi) = \left(\frac{g(h,\xi)\pi^h(\xi)}{\lambda^h(\xi_0)}\right)^{-1}$ for all $h = 2, .., I$ and all $\xi \in Z$. So,

$$\frac{\pi^1(s|\xi)u^{1'}(x_s^1)}{\lambda^1(\xi)} = \frac{\pi^h(s|\xi)u^{h'}(x_s^h)}{\lambda^h(\xi)} \text{ for all } h, \xi \in Z$$

Finally, set $q_k(\xi) = \sum_s \frac{\pi^h(s|\xi)u^{h'}(x_s^h)}{\lambda^h(\xi)} V_s^k$ for all $h, k, \xi \in Z$. These are the FOCs necessary and sufficient for the optimization problem in part (b) of the sequential equilibrium problem of each agent, with $x_s^h(\xi_l) = x_s^h$. Since this implies that $z^h(\xi) = z^h(\xi_0)$ for all $\xi \in Z$, the budget constraints and equilibrium conditions hold trivially. We have just constructed a no retrade sequential equilibrium. Lemma 10 completes the proof.

Necessity Since markets are Θ - complete, we can transform the security payoff matrix into an $S \times S$ identity matrix, without loss of generality. We call this transformed payoff matrix V. By Lemma 11, a no retrade equilibrium is a no retrade sequential equilibrium. Here, the key FOCs from agents' optimization problems in a no retrade sequential equilibrium are $\pi^h(s)u^{h\prime}(x_s^h(\xi_0)) = \lambda^h(\xi_0)q_s(\xi_0)$ for all h, s and $\pi^h(s|\xi)u^{h\prime}(x_s^h(\xi)) = \lambda^h(\xi)q_s(\xi)$ for all $h, s, \xi \in Z$. So,

$$\frac{\pi^h(s)u^{h'}(x_s^h)}{\lambda^h(\xi_0)} = \frac{\pi^j(s)u^{j'}(x_s^j)}{\lambda^j(\xi_0)} \text{ for all } h,\ j,\ s$$

$$\frac{\pi^h(s|\xi)u^{h'}(x_s^h)}{\lambda^h(\xi)} = \frac{\pi^j(s|\xi)u^{j'}(x_s^j)}{\lambda^j(\xi)} \text{ for all } h,\ j,\ s,\ \xi \in Z$$

Dividing these expressions implies that:

$$\frac{\pi^h(s|\xi)\lambda^h(\xi_0)}{\pi^h(s)\lambda^h(\xi)} = \frac{\pi^j(s|\xi)\lambda^j(\xi_0)}{\pi^j(s)\lambda^j(\xi)} \text{ for all } h,\ j,\ s,\ \xi \in Z$$

Recall that $\frac{\pi^h(s|\xi)}{\pi^h(s)} = \frac{\pi^h(\xi|s)}{\pi^h(\xi)}$. So that,

$$\frac{\pi^h(\xi|s)\lambda^h(\xi_0)}{\pi^h(\xi)\lambda^h(\xi)} = \frac{\pi^j(\xi|s)\lambda^j(\xi_0)}{\pi^j(\xi)\lambda^j(\xi)} \text{ for all } h,\ j,\ s,\ \xi \in Z$$

Dividing by the corresponding expression for s' concludes the proof. □

7.2 Proof of Theorem 5

Consider the two-period finance economy where traders trade securities whose payoffs are represented by matrix V. The FOCs necessary that are satisfied in the equilibrium outcome are:

$$q^k(\xi_0)\lambda^h(\xi_0) = \sum_s \pi^h(s)u^{h'}(x_s^h(\xi))V_s^k$$

where $\lambda^h(\xi_0)$ are multipliers. Set $\pi_s^h \equiv \frac{\pi^h(s)u^{h'}(x_s^h(\xi))}{\lambda^h(\xi_0)}$ then $q(\xi_0) = \pi^h V$ where $q(\xi_0)$ is $1 \times J$, π^h is $1 \times S$ and V is $S \times J$. When markets are Θ - complete, $Rank(V) = S$ and $\pi^h = \pi^{h'}$ for all $h, h' \in I$. When markets are Θ - incomplete ($Rank(V) < S$) then it is well-known that there exists a set $\Omega^* \subset \Omega$ whose complement has measure zero such that if $\omega \in \Omega^*$, then $\pi^h \neq \pi^{h\prime}$ for all h, h'. [See for example Theorem 11.6 in Magill and Quinzii (1998).]

Proof of Theorem 5 The argument above shows that if markets are incomplete, there exists a set $\Omega^* \subset \Omega$ whose complement has measure 0, so that if $\omega \in \Omega^*$, then agents' gradients obtained from the first part of the sequential equilibrium problem are different. We prove the theorem by contradiction. Suppose that (x, z, q) is a no retrade equilibrium. Lemma 11 implies that $(x(\xi_0), x, z, q)$ is a no retrade sequential equilibrium. This in turn implies that there exist scalars $\lambda^h(\xi_0) \in \mathbb{R}_{++}$ so that:

$$q^k(\xi_0)\lambda^h(\xi_0) = \sum_s \pi^h(s)u^{h'}(x_s^h)V_s^k$$

Also, there exist $\mu^h(\xi) \in \mathbb{R}_{++}$ for $\xi \in Z$ so that:

$$q^k(\xi)\mu^h(\xi) = \sum_s \pi^h(s|\xi)u^{h'}(x_s^h)V_s^k \tag{5}$$

Define:

$$\pi_s^h \equiv \frac{\pi^h(s)u^{h'}(x_s^h)}{\lambda^h(\xi_0)} \tag{6}$$

By equation 5:

$$\frac{1}{\mu^h(\xi)}\sum_s \pi^h(s|\xi)u^{h'}(x_s^h)V_s^k = \frac{1}{\mu^1(\xi)}\sum_s \pi^1(s|\xi)u^{1'}(x_s^1)V_s^k$$

Using equation 6 yields:

$$\frac{\lambda^h(\xi_0)}{\mu^h(\xi)}\sum_s \frac{\pi^h(s|\xi)}{\pi^h(s)}\pi_s^h V_s^k = \frac{\lambda^1(\xi_0)}{\mu^1(\xi)}\sum_s \frac{\pi^1(s|\xi)}{\pi^1(s)}\pi_s^1 V_s^k$$

But:

$$\frac{\pi^h(s|\xi)}{\pi^h(s)} = \frac{\pi^h(\xi|s)}{\pi^h(\xi)}$$

So that:

$$\frac{\lambda^h(\xi_0)}{\mu^h(\xi)\pi^h(\xi)}\sum_s \pi^h(\xi|s)\pi_s^h V_s^k = \frac{\lambda^1(\xi_0)}{\mu^1(\xi)\pi^1(\xi)}\sum_s \pi^1(\xi|s)\pi_s^1 V_s^k$$

This in turn implies that:

$$\frac{\sum_s \pi^h(\xi|s)\pi_s^h V_s^k}{\sum_s \pi^h(\xi|s)\pi_s^h V_s^j} = \frac{\sum_s \pi^1(\xi|s)\pi_s^1 V_s^k}{\sum_s \pi^1(\xi|s)\pi_s^1 V_s^j}$$

Using Assumption 3 implies that:

$$\frac{\sum_s \pi^h(\xi|s)\pi_s^h V_s^k}{\sum_s \pi^h(\xi|s)\pi_s^h V_s^j} = \frac{\sum_s \pi^h(\xi|s)\pi_s^1 V_s^k}{\sum_s \pi^h(\xi|s)\pi_s^1 V_s^j} \tag{7}$$

From our assumptions, we know there exists a security $K \in \{1, .., J\}$ that pays off in only one state, say $s_K \in \{1, .., S\}$. Taking $j = K$ in the equation above implies:

$$\frac{\sum_s \pi^h(\xi|s)\pi_s^h V_s^k}{\pi_{s_K}^h V_{s_K}^K} = \frac{\sum_s \pi^h(\xi|s)\pi_s^1 V_s^k}{\pi_{s_K}^1 V_{s_K}^K}$$

The price of security K in the first part of the sequential equilibrium satisfies:

$$q_K(\xi_0) = \pi_{s_K}^h V_{s_K}^K = \pi_{s_K}^1 V_{s_K}^K$$

So that:

$$\sum_s \pi^h(\xi|s)\pi_s^h V_s^k = \sum_s \pi^h(\xi|s)\pi_s^1 V_s^k$$

Summing over securities implies that:

$$\sum_s \pi^h(\xi|s)\pi_s^h\left(\sum_k V_s^k\right) = \sum_s \pi^h(\xi|s)\pi_s^1\left(\sum_k V_s^k\right)$$

Using $\pi_{s_K}^h = \pi_{s_K}^1$, and assuming without loss of generality that $S = s_K$, we obtain:

$$\sum_{s=1}^{S-1} \pi^h(\xi|s)\pi_s^h\left(\sum_k V_s^k\right) = \sum_{s=1}^{S-1} \pi^h(\xi|s)\pi_s^1\left(\sum_k V_s^k\right)$$

The vector $\left(\sum_k V_.^k\right)$ lies in $\mathbb{R}_{++}^S$ by our assumption that at least one security pays off in each state. Let $y_s^h = \pi_s^h - \pi_s^1$. We know by assumption that $L \geq S - 1$. In matrix form, the first $S - 1$ equations (by varying ξ) and $S - 1$ unknowns can be written as $M \cdot \left[y_1^h, .., y_{S-1}^h\right]$ where M is an $(S-1) \times (S-1)$ matrix where $M_{i,j} = \pi^h(\xi_i|j)\left(\sum_k V_j^k\right)$. The matrix M has full rank since there exists at least one agent such that the matrix $\{\pi^h(\xi|.)\}_{\xi \in Z}$ is of full rank. So, there exists a unique solution to the first $S - 1$ equations, namely $y_s^h = 0$, $1 \leq s \leq S - 1$. So $\pi_s^1 = \pi_s^h$ for all $s \in \Theta$. This contradicts the fact that that normalized utility gradients are different if $\omega \in \Omega^*$. So, for each $\omega \in \Omega^*$, the resulting equilibrium cannot be a no retrade equilibrium. So, either an equilibrium exists involving retrade, or there is no equilibrium. □

7.3 Proof of Theorem 7

Proof of Theorem 7 By Theorem 4, there exists a no retrade equilibrium (x, z, q). It is also an asymmetric information equilibrium for any information structure since the optimization problem in the asymmetric information case has more constraints and these constraints are satisfied. □

7.4 Proof of Theorem 8

For the proof of Theorem 8, we will need the following.

Definition 12 *Given an economy $\mathcal{E}$, an equilibrium with symmetric information Y is a collection (x, z, q) such that*

(a) $(x^h, z^h) \in \arg\max U^h$ *subject to*

$$\begin{aligned} q(\xi_0)z^h(\xi_0) &= 0 \\ q(\xi)z^h(\xi) &= q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z \\ z^h(\xi) &= z^h(\xi') \text{ for all } \xi, \xi' \in \sigma \in Y \\ x^h(\xi) - \omega^h(\xi) &= V z^h(\xi) \text{ for all } \xi \in Z \end{aligned}$$

(b) $\sum_{h=1}^{I} z^h(\xi_0) = 0,\ \sum_{h=1}^{I} z^h(\xi) = 0,\ \sum_{h=1}^{I} x^h(\xi) - \omega^h(\xi) = 0$ *for all* $\xi \in Z$.

Proof of Theorem 8 By way of contradiction, given $\omega \in \Omega^*$, suppose there exists an equilibrium (x, z, q) with asymmetric information that involves no retrade. Then it must also be an equilibrium with symmetric information Y since the symmetric equilibrium has more constraints and these constraints are satisfied by (x, z, q). Note that if an equilibrium with symmetric information exists, it must involve retrade. Indeed, the proof of Theorem 5 can easily be adapted to the case of symmetric information economies. Simply replace the individual elements of the signal space by the elements of the meet Y. The condition that $\#Y \geq S - 1$ and Assumptions 7 and 6(c) ensure that the proof of Theorem 5 still applies. This, in turn, implies that the equilibrium with symmetric information (x, z, q) must involve retrade. This is a contradiction to our assumption. So, if an equilibrium with asymmetric information exists, it must involve retrade. □

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